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Flexibility valuation in an air transport fleet investment

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DIREITOS DE AUTOR E CONDIÇÕES DE UTILIZAÇÃO DO TRABALHO POR TERCEIROS

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STATEMENT OF INTEGRITY

I hereby declare having conducted this academic work with integrity. I confirm that I have not used plagiarism or any form of undue use of information or falsification of results along the process leading to its elaboration. I further declare that I have fully acknowledged the Code of Ethical Conduct of the University of Minho.

RESUMO

O principal objetivo deste trabalho é a criação de vários modelos para avaliar a flexibilidade de investimentos em frota de linha aérea. Estes modelos estão baseados na filosofia da utilização de opções reais para avaliar o investimento à base de fluxos de caixa, mas também futuras oportunidades, tais como a opção de adiar o investimento e renovar a frota. Pretende-se com isto clarificar a tomada de decisão para investimentos em frota por parte dos transportadores aéreos.

As encomendas de frota de linha têm vindo a aumentar ao longo dos anos e a indústria aeronáutica mostra sinais sólidos de crescimento enquanto meio de transporte eficente de longa distância. Desta forma, é de esperar um aumento do trafego aéreo e com ele um aumento do número de encomendas.

É sabido que o Valor Atualizado Líquido (VAL) é inadequado para valorização de investimentos reais num cenário de incerteza. E, por essa razão, este trabalho utiliza opções reais como forma de avaliação dinâmica, tendo em conta algum tipo de flexibilidade por parte do gestor, ao longo da vida do investimento. A incerteza penaliza os projetos, mas o seu efeito negativo pode ser atenuado ou até mitigado, tomando as decisões certas na altura mais indicada. A abordagem é bastante teórica, mas deixa bem clara a mais valia de olhar para o projeto de uma maneira não estática, isto é, mostra o quão valioso é modular o investimento consoante os principais factores influenciadores do negócio. Quer a opção de investir e a opção de renovar adicionam valor ao projeto, assim sendo, também a opção combinada apresenta um maior valor do que a sua avaliação através de uma análise estática.

Palavras-chave: Flexibilidade; Incerteza ; Investimentos; Opções-reais; Tomada de decisão.

ABSTRACT

The main goal of this work is to create various models to evaluate investment flexibility in air transportation fleet. Every model is based on the Real Options concept to evaluate not only cash flows, but also future opportunities, such as the option to defer the investment and renew the fleet. The purpose is to clarify the decision-making process taken by air transportation managers for investments in fleet.

The orders for airliners have been growing over the past few years and the aerospace industry shows solid signs of growth as mean of transportation for long distances, as so, one can only assume an overflowing air traffic and an increase in the number of orders.

It is known that the Net Present Value (NPV) is inadequate to evaluate real investments in a scenery of uncertainty. For that reason, this work uses real options as a mean of dynamic evaluation, taking into account some degree of flexibility that can be explored by managers, during the lifetime of the investment. Uncertainty penalizes projects, but its negative effects can be attenuated or at best mitigated, by taking the right decisions, at an optimal point in time. This is a theoretical approach that shows the value added by looking at investments in a none static manner. Hence, one modeled the valuation of the project, adapting it to the main drivers of the market. Both the Option to Invest and the Option to Renew add value to the project. Therefore, the combined option value is also higher than its static analysis valuation.

Keywords: Decision-making; Flexibility; Investments; Real Options; Uncertainty.

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INTRODUCTION

The creation of the option models has the major purpose of proving how options add value to projects, these increments depend on the nature of the project and the opportunities that give birth to the option. It is known that every project has its possible future opportunities, and by using any static valuation analysis the value of those opportunities is ignored, under-valuating every project. Therefore in this work, one expects that every option model has a higher value than its underlying project, and as we keep adding options to the portfolio its value keeps increasing.

Real Options Analysis (ROA) is a dynamic method for valuation and provides us with an alternative approach for the tradition valuation methods, such as the Net Present Value (NPV). An option is the right, but not the obligation, to take an action in the future. A real option is the same concept, it is derived from financial options but applied to the real world.

The topic has been vastly studied, and has various approaches depending on the authors, but the concept is always the same, to calculate the present value of projects taking in account for flexibility during the lifetime of the project. The more the flexibility, the more the investment can be reversible and so, new market opportunities will appear and add value to project. Of course these come on an unknown point in time, but the value added by them is still there, along with the decision process, which we assume is made optimum and aligned to maximize the value of the project, as investments are undertaken with an expectation of future profit.

The main principle of the models is based on the fact that the project depends on various factors which one cannot control, that may give birth to new opportunities. These opportunities are classified under the form of options to defer, expand, contract, abandon, suspend, switch or to make a staged investment. Therefore, they should be taken into account in the valuation process, in order to maximize the projects value.

The NPV is the traditional and widely accepted method of valuation of investments, and is based on the concept of comparing the present values of revenues to present values of costs, measuring cash flows over time. Simplistically, if the value is positive one should undertake the project as it is a profitable investment, whether negative it should be rejected. This approach is really handy and easy to use, and maybe that is why it is so popular, but, it sees every project as if it was static, assuming it is managed in a passive way and neglecting the fact that projects are capable of evolving and devolving over time. Furthermore, future cash flows are difficult to predict and vulgarly estimations are made on a defensive way to avoid unprofitable projects.

If one wants to evaluate the possible adjustments in a project, an ROA should be used, taking into consideration some degree of flexibility of the project. This flexibility is seen as a decision taken sometime along the life of the project, only once or multiple times. Lattices are, on most cases, the more visual and explicit manner of describing the branches of decision of the project.

On the other hand, applying it with real world variables and wanting to evaluate all the possible opportunities is impracticable as complexity thrives at each input added, turning it out almost impossible. As so, for simplification purposes, one will consider only one factor models, and a combination of two options. The main driver of the market is assumed to be the price of tickets (P), a stochastic variable, and costs (C) are constant along with other market indicators. Later, in order to address reality, the variables P and C will change its meaning, along with some of the other constants.

In real life the number of opportunities and paths a project can follow are far too many to count, so the perfect evaluation will never be done as there is always place for change in real life projects. Nevertheless, the goal is to always find the best estimation, for a more informed decision. Most of the times the aviation industry is seen as a cutting-edge technology sector, but in fact not much has changed on the industry over the last 50 years. The second world war brought new technologies, and those were put to work onto commercial aviation, but we had supersonic transatlantic flights and now we do not. The industry is focused on profit and on improving the safety and efficiency of the actual technology, instead of finding a new revolutionary mean of air navigation, as research and development is very costly in such competitive environment.

The aviation industry is a growing as the number of commercial flights and the number of ordered airliners is rising over the last years, aside from some outliers caused by some episodic event, such as last year's Boing 737 Max grounding or this year's pandemic COVID-19. Financially, the industry is a growing market, and seems to follow most of the world economic cycles, as it depends on the price of oil and the purchasing power of the consumers.

More disruptive ideas aside, on a stable yet cyclical industry which aims to improve its efficiency, one can suppose that the more valuable factor to take in account for an acquisition of fleet is when we invest, and the further possibility to update our fleet to a more efficient technology. That is why this work will focus on the option to defer the investment, and a specific switch option, the investment in a new aircraft, salvaging the old one. Furthermore, it merges these first two options, modeling a project whose aircraft can be replaced by a more modern one and which the initial investment can be deferred, meaning an option with an embedded option.

On section 3.1, the fundamentals of the mathematical models will be presented. These serve as base constructs for the options' models, along with original models for option valuing. Section 3.2 and 3.3 will focus on the Option to Invest and the Option to Renew, respectively, and every aspect about the adaptation of the models to the type of options chosen. It has been proved that both add value to the project and therefore should be accounted for during the valuation process. Section 3.4, and main goal of this project focuses on valuating the option of investing with an embedded Option to Renew the fleet. This option model is a combination of the previous models, every step of its construction is described on the chapter and leads us to a conclusion that it also adds value to the project and to each simple option, as these increments of value are different depending on the investment opportunity. In each option, one can find the optimum time at with each option should be exercised, depending on the drivers of the market. Chapter 4 addresses values to each constant parameter and constructs base case scenarios, along with an extensive analysis on how market variables change the option and project value. At its end, there is a brief summary of the numerical sensibility analysis made to each model.

At last, but before summing all into a conclusion, on chapter 5, there is a reflection on the work done and its relevance to the industry. To address the actual state of the industry another interesting business opportunity, the suspension of operations, is proposed to be modulated in the future, along with the options presented on this work.

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LITERATURE REVIEW

Graham and Harvey (2001) and Gibson and Morrell (2005) found that airline managers use majorly Discounted Cash Flow (DCF) techniques, such as Net Present Value (NPV) and Internal Rate of Return (IRR) to evaluate investments. These techniques are of great use, as they are easy to use and understand, although they miss important information, and a simple rule as, invest if NPV is positive, will not produce a totally informed decision.

The traditional valuation methods rely on discounted cash flows, and that raises several potential problems Mun (2006). NPV and all other income based approaches undervalue assets that reward more on early stages, as liquidation is not taken into account. Furthermore, cash flow streams are really difficult to estimate, as the economic life of the investment is unknown, and forecasting errors on estimations are inevitable. As the analysis produces a positive or negative result, which will be responsible for a ”go” or ”no go” decision, it is seen as being over simplistic and misleading. Other valuation methods, that use the weighted average cost of capital (WACC) or comparables, also suffer from some of the previous flaws or other particular to their own concept that, not only skew the results, but also orient the manager to think in a specific manner. In contradiction, the manager should have a broad overview of the project, and that is way any on these should be seen individually.

All traditional approaches have their advantages and disadvantages, but when in the comparison to Real Options, all these analyses previously mentioned, only consider one course of action, leading project managers to dismiss any degree of flexibility during the project (Copeland and Antikarov, 2001). Nowadays, in this rapid changing and highly fluid market environment, every manager is supposed to shape its business to maximize the wealth of the company. As we assume that every manager has the responsibility of doing so, that ability should be valuated in the first place and traditional approaches would grossly under valuate every investment opportunity.

An option is the right, but not the obligation, to take an action in the future (Amram and Kulatilaka, 1999). The real options concept, brought to light by Myers (1977), derives from a financial option, adapted to be a real world opportunity. The opportunities can be varied depending on the nature of the project we are evaluating. For example, if we are building a mining project, the investment is partially or completely irreversible, as there is no salvage value for the plant and it is very specific to be used for any other purpose. In this project the option to abandon (Robichek and Van Horne (1967) along with Dyl and Long (1969), and later in McDonald and Siegel (1985) and Myers and Majd (2001)), seems out-ruled, yet, there are other sorts of possibilities as the option to suspend, (Dixit et al., 1994), the option to expand or contract (Dixit (1988) and Trigeorgis (1993)), the Option to Invest (Tourinho et al. (1979); McDonald and Siegel (1986); Majd and Pindyck (1987); Carr (1988) and Paddock et al. (1988)). For a more general discussion, as the time to invest is in most cases a choice from the investor, the Option to Invest is an option that can add value to almost every project, as timming the market is very important. McDonald and Siegel (1986) studied the value of waiting and found out that, for some parameter, the optimal entry is at a time benefits are twice the investment costs, and not after passing the threshold when they are equal, $NPV = 0$.

The variety of options a project can undergo and the timing they can be executed, creates a branched tree of decision to be taken by manager to maximize the projects value. These options are the managers response to uncertainty on the market, and therefore, of great value. Valuing the project with those decisions embedded is not, in most cases, the difficult part. What brings complexity is the fact that those options are to be exercised in the future, depending on variables which are yet unknown. To better estimate the value of an investment, we will assume those unknown variables are stochastic and follow a simple random walk, as the referred on Cox and Miller (1977) or another stochastic process relevant to the variables behaviour. More exhaustive studies on these processes may include Merton (1973), Feller (2008), Karlin (1975) and later Karlin and Taylor (1981). These processes are distinguishable, as Insley (2002) concluded that the option value and the optimal exercise time are significantly different under the mean reversion process, when compared to a geometric Brownian motion. In this study we assume that the stochastic variable follows a geometrical Brownian motion.

Quantitative approach to Real Options descend from Black and Scholes (1973) and Merton (1973) and arrived from the need to bring uncertain cash flows to the present value. These valuation method is based on Bachelier (1900), whose model is then extended by the incorporation of the CAPM model by Samuelson (2015). There are three types of approaches to value an option: The use of lattices as in Trigeorgis (1991); Simulations of the stochastic process, as the Least Squares Monte-Carlo method, initially used by Boyle (1977); and the use of the analytical solution of the differential equations that describe the stochastic process, which we will proceed with in this work.

Some years later, plenty of studies come up using RO's to valuate infrastructure investments, as a tool to attenuate uncertainty, taking in account for all the possible outcomes. But, in many cases, the sources of uncertainty in the project are state variables, which are not traded assets (Schwartz and Trigeorgis, 2004), turning out difficult to modulate through a wiener process or a mean reverting process. If these variables are also path dependent, the simpler way is to predict them, using Monte Carlo simulations as in Longstaff and Schwartz (2001), which estimates the continuation value by a least squares regression. This technique is being refined through time, from various authors such as Boyle et al. (2001), catch up with some methods for less time consuming simulations. Further tests to the accuracy of several variance reduction techniques by Areal et al. (2008) found that using low discrepancy sequences can improve its accuracy.

Concerning the differential equations method, one of the most well known books is "Investments under uncertainty" from Dixit et al. (1994), which stand on some of the neoclassical financial theory assumptions (see also Majd and Pindyck (1987)). Here, various options and their optimal exercise are analysed, because as a contingency claim approach, it is based on the idea that investors want to maximize the value of the investment and optimally time the investment. At the end of the book, some more advanced extensions to those models are approached, along with its applications. "Renewing Assets with Uncertain Revenues and Operating Costs", from Adkins and Paxson (2011), and "Replacement decisions with multiple stochastic values and depreciation", Adkins and Paxson (2017), are the most comparable researches with mine, in which a two factor renewal option model produces a quasianalytical solution. Studies found that after the replacement, an increase on revenues has a larger influence on the boundary than either the operating cost or the renewal cost. More recent result point that an increase in operating costs volatility defers the replacement and increases the value of the deferral, and that the presence of a salvage value and tax depreciation significantly lowers the operating costs threshold. Furthermore, it states that the actual optimal replacement depends on which factors are uncertain, so understanding the operational context is critical to making a proper decision. In my case, I will assume that revenues are uncertain, hence I will be using only a one factor model, a stochastic variable which follows a geometrical Brownian motion.

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THE MODELS

3.1 VALUING PROJECTS

In order to find the value of an option, one must learn how to evaluate projects, which in our case is an investment in air transportation fleet. In order to find the investment value, one has to find the drivers of the market that at a great scale influence our investment. In a general stand point the drivers of almost any market are its costs and revenues. As the purpose is to work with one stochastic variable, the operating costs will be held constant and the uncertainty will arrive from the revenues, P. This variable is assumed to follow a geometrical Brownian motion (GBM), Which is a stochastic process with the following properties:

- Future values only depend on the current value and not on previous values - Markov process;
- It has independent increments - Wiener process;
- These increments are normally distributed, and its volatility increases linearly over the time interval.

A Brownian motion with drift can be represented by:

$$dP = \alpha P dt + \sigma P dz \tag{1}$$

This simple wiener process is composed by the profit flow in perpetuity, where α is the drift parameter, along with increments to the current value, dz, with a variance parameter of σ. Then, repeating Dixit et al. (1994), $V(P)$ is the solution to the investment value as a function of its primary output, P, and is expressed by the following equation 2.

$$\frac{1}{2}\sigma^2 P^2 V''(P) + (r-\delta)PV'(P) - rV(P) + \Pi(P) = 0 \tag{2}$$

$$Q \equiv \frac{1}{2}\sigma^2 \beta(\beta-1) + (r-\delta)\beta - r = 0 \tag{3}$$

As the homogeneous part of the equation (2) has a solution of the form $V(P) = BP^{\beta}$, and provided that the β's are the roots of the quadratic equation on (3), with some assumptions, it is known that the homogeneous solution is a linear combination of two independent solutions, for example, $B_1 P^{\beta^1} + B_2 P^{\beta^2}$, plus a solution to the non-homogeneous part, the profit flow $\Pi(P)$. Leaving us with:

$$V(P) = B_1 P^{\beta_1} + B_2 P^{\beta_2} + \Pi(P) \tag{4}$$

Where, from the roots of the quadratic equation (3) we find the values of β_1 and β_2, which only depend on variables of the market r, δ and σ.

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{5}$$

$$\beta_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0 \tag{6}$$

The non-homogeneous part of the equation (2), $\Pi(P)$, is different depending on the values of P and C, as one will stay inactive if $P < C$ and only start to operate if $P > C$, leaving us with $\Pi(P) = max[P - C, 0]$. Solving the equation to both regions, we get if $P < C$ that $\pi(P) = 0$ and by adding $\frac{P}{\delta} - \frac{C}{r}$ to the homogeneous solution and substituting we can see that it satisfies the solution, so when $P > C$, $\Pi(P) = \frac{P}{\delta} - \frac{C}{r}$. Summing it all, we end up with:

$$V(P) = \begin{cases} K_1 P^{\beta_1} + K_2 P^{\beta_2} & P \leq C \\ B_1 P^{\beta_1} + B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{r} & P > C \end{cases} \tag{7}$$

As we have two regions, one where P can be equal to 0 and another where P can go to infinity, we have to impose limits to the behaviour, otherwise the results would be illogical. When P assumes values close to 0 the option value will be close to zero too.

As $\lim_{P \to 0} P^{\beta_2} = \infty$ we have to neuter that P^{β_2} with an absorbing coefficient, therefore $K_2 = 0$. On the other branch as we need to rule out the positive power of P^{β_1} to avoid speculative bubbles, we neuter it setting $B_1 = 0$. At $P = C$, using value matching and smooth pasting at C, we have:

$$K_1 C^{\beta_1} = B_2 C^{\beta_2} + \frac{C}{\delta} - \frac{C}{r} \tag{8}$$

$$\beta_1 K_1 C^{\beta_1} = \beta_2 B_2 C^{\beta_2} + \frac{C}{\delta} \tag{9}$$

Using equations (8) and (9) one finds the values of the coefficients K_1 and B_2 to be:

$$K_1 = \frac{C^{1-\beta_1}}{\beta_1 - \beta_2} \left(\frac{\beta_2}{r} - \frac{\beta_2 - 1}{\delta} \right) \tag{10}$$

$$B_2 = \frac{C^{1-\beta_2}}{\beta_1 - \beta_2} \left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{\delta} \right) \tag{11}$$

Knowing this, we are able to define the value of the project as:

$$V(P) = \begin{cases} K_1 P^{\beta_1} & P \leq C \\ B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C}{r} & P > C \end{cases} \tag{12}$$

3.2 OPTION TO INVEST

With the value of the project correctly expressed, the aim is to model the Option to Invest and find the value of P that triggers the exercise of the option, maximizing the value of the project. As suggested by Dixit et al. (1994), one can easily find the value of the Option to Invest, using the $V(P)$ as the underlying value, at a determined exercise price, P^*, the value of the option will be equal the value of the active project minus the investment.

As before, one can use the ODE solution, whose homogeneous part's solution is a linear combination of two solutions of the type AP^{β}, this time changed the coefficients in order not to mistake it by the previous K and B. As so, the options value is of the type:

$$F(P) = A_1 P^{\beta_1} + A_2 P^{\beta_2} \tag{13}$$

When P assumes values close to 0, the option value will be close to 0 too, as $\lim_{P \to 0} P^{\beta^2} = \infty$ we assume that A_2 is 0. On the hand, when $P > P^*$ the value of the investment opportunity is defined for the value of the operating project, and so matches the investment contingencies. Hereby, the option value is defined by:

$$F(P) = \begin{cases} A_1 P^{\beta_1} & P \leq P^* \\ V(P) - I & P > P^* \end{cases} \tag{14}$$

I is a constant which stands for the investment made to start operating, and besides that no variable is new here. As we know that $P^* > C$, otherwise it would not make sense as we are paying to start operating and not to stay waiting, we know the region of V(P) we are working with, $P > C$, so $V(P)$ can be substituted by $B_2 P^{\beta^2} + \frac{P}{\delta} + \frac{C}{r}$. At the boundary condition, P=P*, we have:

$$F(P^*) = V(P^*) - I \tag{15}$$

$$A_1 P^{*\beta_1} = B_2 P^{*\beta_2} + \frac{P^*}{\delta} - \frac{C}{r} - I \tag{16}$$

Assuming a smooth pasting condition which is to be expect, we have:

$$F'(P^*) = V'(P^*) \tag{17}$$

$$\beta_1 A_1 P^{*\beta_1} = \beta_2 B_2 P^{*\beta_2} + \frac{P^*}{\delta} \tag{18}$$

With the last two pairs of equations one yields the value of A_1 and P^*, which only depend of I and β_1 that in turn depends on the constants r,δ, and σ:

$$P^* = \frac{\beta_1}{\beta_1 - 1} \delta I \tag{19}$$

$$A_1 = \left(\frac{P^*}{\delta} - \frac{C}{r} - I \right) \left(\frac{1}{P^*} \right)^{\beta_1} \tag{20}$$

Ending up with an investment opportunity value equal to:

$$F_i(P) = \begin{cases} \left(\frac{P^*}{\delta} - \frac{C}{r} - I \right) \left(\frac{P}{P^*} \right)^{\beta_1} & P \leq P^* \\ V(P) - I & P > P^* \end{cases} \tag{21}$$

3.3 OPTION TO RENEW

Let's now assume that sometime along the life of the project, we have the option to replace our aircraft by a more efficient one. This implies that the airliners are directly comparable, they have the same number of seats, etc, so that both can reward us with the same revenues. Except for the costs, which are different, the cost of operating the new aircraft is lower than the cost of the older one, $C_2 < C_1$.

Again, a mathematical model was constructed to find the optimum time at which one should exercise the Option to Renew fleet. The principles and assumptions will be the same as the ones used in the Option to Invest, aside for 3 new variable: C_2 which stands for the cost of operating the new aircraft; C_1 representing the cost of operating the old aircraft; and R which is the price of the renewal. The methodology for evaluating the project is maintained, so again, the homogeneous option value is still a linear combination of two solutions AP^{β}, plus a profit flow. The difference is that now there are two projects, one for the older aircraft $V(P, C_1)$ and one for the new one $V(P, C_2)$. Each project value is represented by:

$$V_r(P,C) = \begin{cases} K_1(C)P^{\beta_1} & P \leq C \\ B_2(C)P^{\beta_2} + \dfrac{P}{\delta} - \dfrac{C}{r} & P > C \end{cases} \tag{22}$$

This project value is the same as the used on the previous model, therefore its coefficients will be found equating (8) and (9). As so K_1 and B_2 are expressed as:

$$K_1 = \frac{C^{1-\beta_1}}{\beta_1 - \beta_2}\left(\frac{\beta_2}{r} - \frac{\beta_2 - 1}{\delta}\right) \tag{23}$$

$$B_2 = \frac{C^{1-\beta_2}}{\beta_1 - \beta_2}\left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{\delta}\right) \tag{24}$$

The investment opportunity to replace will have two regions: one where the option is not exercised and its value is represented by a solutionAP^{β}; and a second, where one should exchange one project for the other paying the renewal cost. These two regions are bounded on the trigger price, P_r^*, at which one should replace the aircraft. This yields:

$$F_r(P) = \begin{cases} D_1 P^{\beta_1} & P \leq P_r^* \\ V(P, C_2) - V(P, C_1) - R & P > P_r^* \end{cases} \tag{25}$$

At the boundary condition, $P = P_r^*$, both expressions encounter each other at a smooth pasting condition, meaning that the option value and its derivatives are continuous, and so we equate:

$$D_1 P_r^{*\beta_1} = V(P_r^*, C_2) - V(P_r^*, C_1) - R \tag{26}$$

$$\beta_1 D_1 P_r^{*\beta_1 - 1} = V'(P_r^*, C_2) - V'(P_r^*, C_1) \tag{27}$$

Taking in account that $V(P, C)$ takes different forms depending on the value of P, and $C_2 < C_1$, one can easily define 3 regions: $P \leq C_2$; $C_2 < P \leq C_1$; $P > C_1$.

When $P \leq C_2$, then the value matching and the smooth pasting, leave us respectively:

$$D_1 P_r^{*\beta_1} = K_1 \left(C_2\right) P_r^{*\beta_1} - K_1 \left(C_1\right) P_r^{*\beta_1} - R \tag{28}$$

$$\beta_1 D_1 P_r^{*\beta_1 - 1} = \beta_1 K_1 \left(C_2\right) P_r^{*\beta_1 - 1} - \beta_1 K_1 \left(C_1\right) P_r^{*\beta_1 - 1} \tag{29}$$

As on the smooth pasting condition β_1 and $P_r^{*\beta_1 - 1}$ appear on all members, there is no solution to the equation, meaning no P_r^* is found. So when transferred to reality, if the value of the revenues is lower than the operational cost of both aircraft, none of the projects will be profitable, and there is no P at which the aircraft should be renewed. When $C_2 < P \leq C_1$, we assume the new aircraft would start operating and the old one is still waiting, then the value matching and the smooth pasting, leave us respectively:

$$D_1 P_r^{*\beta_1} = B_2 \left(C_2\right) P_r^{*\beta_2} + \frac{P_r^*}{\delta} - \frac{C_2}{r} - K_1 \left(C_1\right) P_r^{*\beta_1} - R \tag{30}$$

$$\beta_1 D_1 P_r^{*\beta_1} = \beta_2 B_2 \left(C_2\right) P_r^{*\beta_2} + P_r^* / \delta - \beta_1 K_1 \left(C_1\right) P_r^{*\beta_1} \tag{31}$$

Through 30 and 31, we can arrive to the following equation, whose solution is the renewal trigger price, which we will call $P^*_{r,2}$:

$$(\beta_1 - \beta_2)\, B_2(C_2) {P^*_{r,2}}^{\beta_2} + (\beta_1 - 1)\, \frac{P^*_{r,2}}{\delta} - \beta_1 \left(\frac{C_2}{r} + R \right) = 0 \tag{32}$$

As the trigger price cannot be isolated on this solution, previous equation 32 needs to be solved numerically to find the trigger price in region $C_2 < P \leq C_1$.

At the last region when $P > C_1$, then both projects are operating, this yields:

$$D_1 {P^*_r}^{\beta_1} = B_2\,(C_2)\, {P^*_r}^{\beta_2} + \frac{P^*_r}{\delta} - \frac{C_2}{r} - \left(B_2\,(C_1)\, {P^*_r}^{\beta_2} + \frac{P^*_r}{\delta} - \frac{C_1}{r} \right) - R \tag{33}$$

$$\beta_1 D_1 {P^*_r}^{\beta_1} = \beta_2 B_2\,(C_2)\, {P^*_r}^{\beta_2} + \frac{P^*_r}{\delta} - \left(\beta_2 B_2\,(C_1)\, {P^*_r}^{\beta_2} + \frac{P^*_r}{\delta} \right) \tag{34}$$

Equating these two expressions, we have the other solution for the renewal trigger price, which we will call $P^*_{r,3}$:

$$(\beta_1 - \beta_2)\,(B_2(C_2) - B_2(C_1))\, {P^*_{r,3}}^{\beta_2} - \beta_1 \left(\frac{C_2 - C_1}{r} + R \right) = 0 \tag{35}$$

Contrary to the trigger price for the last region, equating this in order to the trigger price we can find the following solution:

$$P^*_{r,3} = \left(\frac{\beta_1 \left(\frac{C_2 - C_1}{r} + R \right)}{(\beta_1 - \beta_2)(B_2(C_2) - B_2(C_1))} \right)^{\frac{1}{\beta_2}} \tag{36}$$

Previous two solutions 32 and 35 can be used to find the the boundary where they meet, at $P = C_1$, as their values will be the same. If we equate both one finds the value of R that triggers each solution.

$$\left[R^* = \frac{C_1 - C_2}{r} + \frac{(B_2(C_1) - B_2(C_2))\, C_1^{\beta_2}\, (\beta_2 - \beta_1)}{\beta_1} \right] \tag{37}$$

Meaning that:

$$P_r^* = \begin{cases} P_{r,2}^* & R \leq R^* \\ P_{r,3}^* & R > R^* \end{cases} \tag{38}$$

With the trigger price defined, it is possible to advance to the renewal opportunity valuation. In the region where the option is exercised, the value of it is easily demonstrated by the exchange of the old project by the new one, paying for the upgrade of aircraft. The region when the option is "out of the money", we have a solution $D_1 P^{\beta_1}$. When $P = P_r^*$, we are at the boundary of both region from system 25 and so:

$$D_1 P_r^{*\beta_1} = V(P_r^*, C_2) - V(P_r^*, C_1) - R \tag{39}$$

Equating this we have:

$$D_1 = (V(P_r^*, C_2) - V(P_r^*, C_1) - R) \left(\frac{1}{P_r^*}\right)^{\beta_1} \tag{40}$$

Expressing D_1 as in equation 40 we can express $D_1 P^{\beta_1}$ in a different form. Leaving us with an option value of the following form:

$$F_r(P) = \begin{cases} (V(P_r^*, C_2) - V(P_r^*, C_1) - R) \left(\frac{P}{P_r^*}\right)^{\beta_1} & P \leq P_r^* \\ V(P, C_2) - V(P, C_1) - R & P > P_r^* \end{cases} \tag{41}$$

3.4 Option to Invest and Renew

With the previous modeled options, we can more easily create a model to include in our decision the Option to Invest and further replace our aircraft. In order to renew an aircraft, we must have one to be substituted, meaning that we have to invest on a first project and only then replace it. Also, in this option model, it is assumed that at the time of the first investment, the technology for the second one is already available, but for some reason the investor cannot buy it. To mode both options to invest and renew combined, one must go backwards, and start with what will happen last. As we first need to invest and only then renew, lets focus on the renewal. When the Option to Renew is out of the money, $P < P_r^*$, its value is defined by $D_1 P^{\beta 1}$, that captures the future possibility of exercising the option. On the Option to Renew, a combination of equation 40 and 41, leads us to express the previous as:

$$D_1 P^{\beta_1} = (V(P_r^*, C_2) - V(P_r^*, C_1) - R)\left(\frac{P}{P_r^*}\right)^{\beta_1} \tag{42}$$

The new Option to Invest will have embedded the possibility of replacing the aircraft, anytime during the life of the investment. So, at each branch of the project value presented in equation 12, we have to sum the previous expression 42. Leaving us with a project value:

$$V_c(P) = \begin{cases} K_1 P^{\beta_1} + (V(P_r^*, C_2) - V(P_r^*, C_1) - R)\left(\dfrac{P}{P_r^*}\right)^{\beta_1} & \text{for } P \leq C_1 \\ B_2 P^{\beta_2} + \dfrac{P}{\delta} - \dfrac{C_1}{r} + (V(P_r^*, C_2) - V(P_r^*, C_1) - R)\left(\dfrac{P}{P_r^*}\right)^{\beta_1} & \text{for } P > C_1 \end{cases} \tag{43}$$

The value of the project do invest with renewal possibility is different from the simple project to invest, although, the coefficients K_1 and B_2 will be the same as we are changing both branches of the equations, canceling each other. At $P = C$, for the value matching and smooth pasting to occur, both branches and its derivatives should equate, yielding us the same coefficients as in equations 10 and 11, which are the following:

$$K_1 = \frac{{C_1}^{1-\beta_1}}{\beta_1 - \beta_2}\left(\frac{\beta_2}{r} - \frac{\beta_2 - 1}{\delta}\right) \tag{44}$$

$$B_2 = \frac{{C_1}^{1-\beta_2}}{\beta_1 - \beta_2}\left(\frac{\beta_1}{r} - \frac{\beta_1 - 1}{\delta}\right) \tag{45}$$

With the values of the coefficients K_1 and B_2 defined, one follows the same process as the proceeded on the Option to Invest, to find the optimum time to invest, but this time the equations' branches are not the same. The Option to Invest and Renew is the opportunity of future investing in the project previously defined, at a given time, depending on P. Therefore, the option can be defined by an homogeneous solution when waiting to exercise, and the activation of the project paying the investment price, as shown:

$$F_c(P) = \begin{cases} A_1 P^{\beta_1} & \text{for } P \leq P_c^* \\ B_2 P^{\beta_2} + \frac{P}{\delta} - \frac{C_1}{r} + (V(P_r^*, C_2) - V(P_r^*, C_1) - R)\left(\frac{P}{P_r^*}\right)^{\beta_1} - I & \text{for } P > P_c^* \end{cases} \tag{46}$$

Again, at the trigger price of this option, when $P = P_c^*$, the value and the derivatives of the branches has to be the same to comply with the value matching and smooth pasting condition. Leading us to the following pair of equations:

$$A_1 P_c^{*\beta_1} = B_2 P^{\beta_2} + \frac{P_c^*}{\delta} - \frac{C_1}{r} + (V(P_r^*, C_2) - V(P_r^*, C_1) - R)\left(\frac{P_c^*}{P_r^*}\right)^{\beta_1} - I \tag{47}$$

$$\beta_1 A_1 P_c^{*\beta_1} = \beta_2 B_2 P_c^{*\beta_2} + \frac{P_c^*}{\delta} + \beta_1 (V(P_r^*, C_2) - V(P_r^*, C_1) - R)\left(\frac{P_c^*}{P_r^*}\right)^{\beta_1} \tag{48}$$

Equating these two, one finds the solution for the trigger price of the option, called P_c^*. This solution is very similar to the boundary solution of the previous options, as $D_1 P^{\beta_1}$ disappears on both sides of the equation. Leaving the solution equal to the Option to Invest, except now the trigger price has another meaning added, it is the optimum price to invest with the further possibility to replace the apparatus.

$$EQ \rightarrow (\beta_1 - \beta_2)\, B_2 P_c^{*\beta_2} + (\beta_1 - 1)\, \frac{P_c^*}{\delta} - \beta_1 \left(\frac{C_1}{r} + I\right) = 0 \tag{49}$$

The trigger price for the combined option is established, however if the price of renewal, R, is too low or the cut in costs is too significant, we have the trigger price of the renewal, P_r^*, lower than the trigger price of the investing on the combined option, P_c^*. Leading to a premature substitution of the aircraft, without having invested on the substituted aircraft in the first place. It can happen for both options to be exercised simultaneously, but never the Option to Renew first. This is ruled out by making sure that a certain set of constant values obey to a certain condition that gets us a $P_c^* \leq P_r^*$. The trigger price of the renewal is a combination of two solutions $P_{r,2}^*$ and $P_{r,3}^*$, as seen on equation 38. As the project value is defined by equation 43, we assume that the investment will only happen on region $P > C$, therefore the solution applied for the trigger price of the renewal is the combination of equations 33 and 34, represented by $P^*_{r,3}$, resulting in equation 36. As so, in order for the trigger price of the combined option to be lower than the option to replace, it should comply with the following:

$$P_c^* \leq \left(\frac{\beta_1 \left(\frac{C_2 - C_1}{r} + R \right)}{(\beta_1 - \beta_2)(B_2(C_2) - B_2(C_1))} \right)^{\frac{1}{\beta_2}}, R > R^* \tag{50}$$

4

NUMERICAL ANALYSIS AND COMPARATIVE STATICS

4.1 BASE CASE

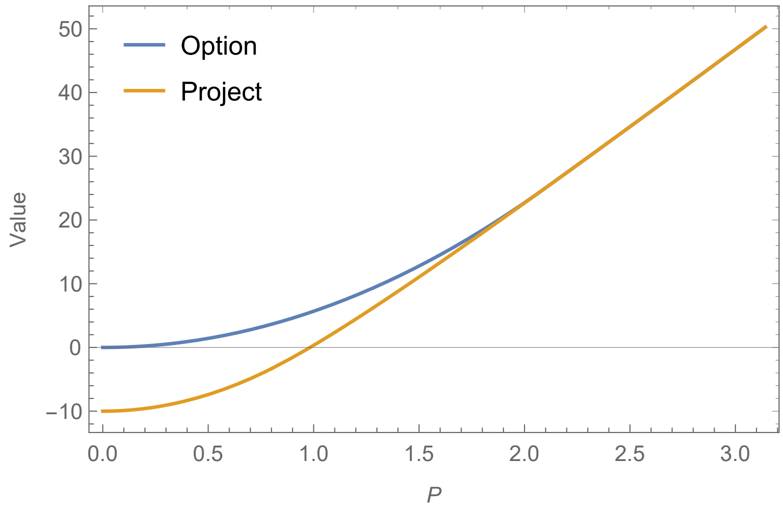
With all the opportunities valued, one can step to a numerical analysis, to better analyse the influence of each parameter on the trigger price, the option and project value. To do so one will assume this set of values present on the following table 1.

Parameter	Description	Value
r	Risk-free rate	0.04
σ	Volatility	0.2
δ	Return Shortfall	0.04
Investment specific		
I	Investment	10
C	Operational cost	0.8
P^*	Investment trigger price	2.09443
Renewal specific		
R	Renewal cost	5.5
C_1	Operational cost of old aircraft	0.8
C_2	Operational cost of new aircraft	0.5
R^*	Renewal cost boundary	1.4
P_r^*	Renewal trigger price	2.4375
Compound specific		
P_r^*	Renewal trigger price	2.4375
P_c^*	Investment trigger price	2.09443

Table 1: Base case parameters.

4.1.1 *Option to Invest*

Figure 1 represents the value of the project and Option to Invest, where the smooth pasting condition of the functions is clearly visible along with the value added by the deferral option to a simple investment process.



$r = 0.04; \delta = 0.04; \sigma = 0.2; I = 10; C = 0.8; P^* = 2.09443$

Figure 1: Value of the project and the Option to Invest.

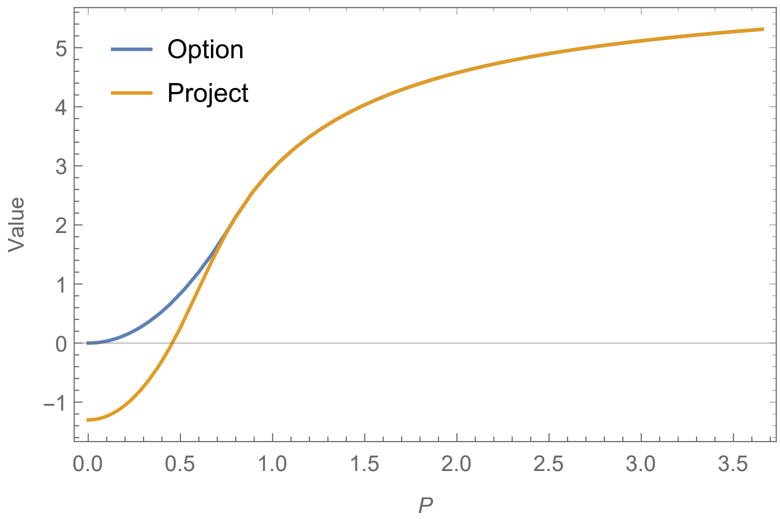
This graph is proof that a standard NPV approach neglects the power of timing the market. On a standard NPV approach, one enters the market earlier, because $\frac{P}{\delta} - \frac{C}{r} - I > 0$ when $P \approx 1$. On the other hand, the option should only be exercised at $P^* = 2.09443$, meaning that the holder of an option starts operating later. Although, the option holder does not have a real profit when P is between the root of the project ($NPV = 0$) and the P^*, his portfolio is more valuable than an invested one, because he is not exposed to a down fall on P.

Assuming most managers have the power and knowledge to choose when to start operating, they will most probably wait until a certain threshold is passed, so that they have a margin to feel safe, even if not the optimum, so that if the price goes down they will still be profiting. As in real life, on a normal standpoint, managers have the option to postpone their entry on the market, it would makes sense to evaluate it. Taking these opportunities into account when evaluating the project as a thought investment and not only valuing a series of cash flows.

At a sum, not investing at a given time and having the possibility to wait and invest in function of the value of P, adds value to the project and therefore to the company.

4.1.2 *Option to Renew*

Assuming these base values, one can find the renewal cost that defines the regions of P_r^*, as being $R^* = 1.4$. Recall that this is the value of R that separates the use of the solutions $P_{r,2}^*$ or $P_{r,3}^*$, that are the boundary conditions for the options to be exercised with different values of C as each aircraft, old and new, have different operating costs. Assuming a renewal cost of $R = 5.5$, we can find a $P_{r,3}^* = 2.4375$. With these inputs we can produce and better analyze the following figure 2.



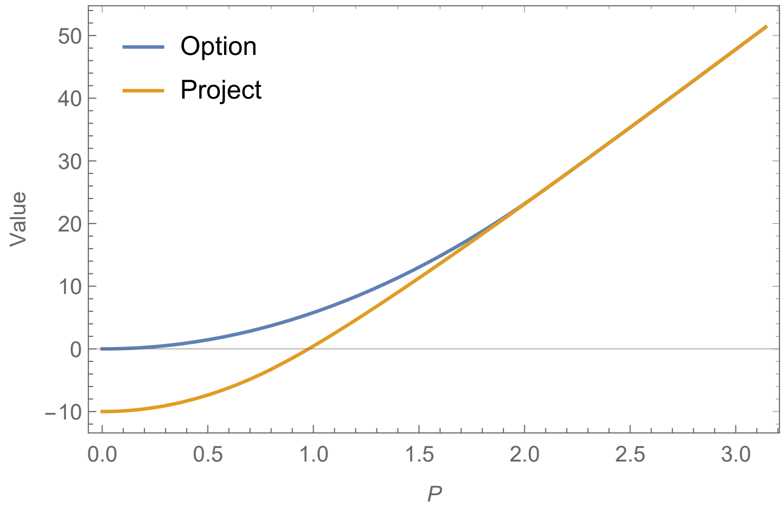
$r = 0.04$; $\delta = 0.04$; $\sigma = 0.2$; $C_1 = 0.8$; $C_2 = 0.5$; $R = 5.5$; $P_{r,3}^* = 2.4375$

Figure 2: Value of the project and the Option to Renew.

Again, the added value by the option is represented by the displacement between the curves when $P < P_r^*$. Take into consideration that we are comparing the Option to Renew with the value of the project of renewing. Further and for a more interesting debate, the value of the option will be combined with an Option to Invest, meaning that the option to replace an aircraft will be taken into account at the time of the investment. Comparing the Option to Invest and further possible renewal and its underlying project will be more enlightening, still, this is an interesting analysis and serves as base for what is to come.

4.1.3 *Option to Invest and Renew*

With these base inputs we are able to plot the value of the project and the Option to Invest with a later opportunity to renew, as a function of P. By doing so, one ends up with the following:



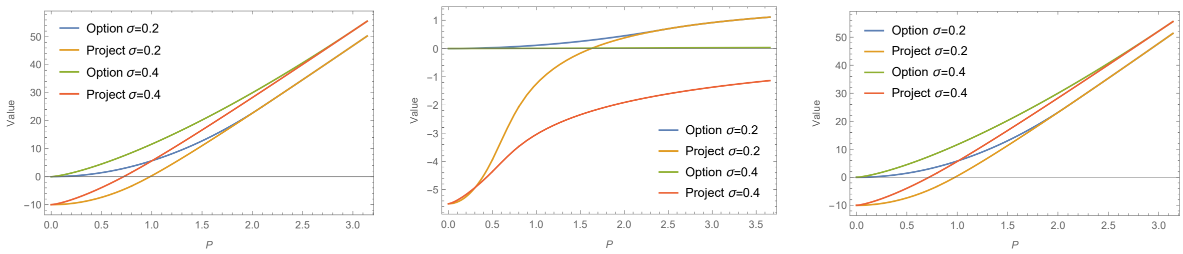
$r = 0.04$; $\delta = 0.04$; $\sigma = 0.2$; $C_1 = 0.8$; $C_2 = 0.5$; $I = 10$; $R = 5.5$; $P_r^* = 2.4375$;$P_c^* = 2.09443$

Figure 3: Value of the project and Option to Invest and Renew.

The previous figure 3 is very identical to figure 1, as they both represent an option Invest, but also this one comprises the possibility of replacing the aircraft for a more efficient one in the future. As the nature of the solutions is the same, using some determined set of values one can make them look exactly the same, by changing the attractiveness of the Option to Renew using values of C2 really close to C1 and high values of R. Anyway, later on this chapter, we will see how the difference in efficiency and the price to pay for the renewal affects the value of the project and the option.

4.2 PARAMETERS

4.2.1 *Volatility* - σ



(a) $V_i(P); F_i(P)$ (b) $V_r(P); F_r(P)$ (c) $V_c(P); F_c(P)$

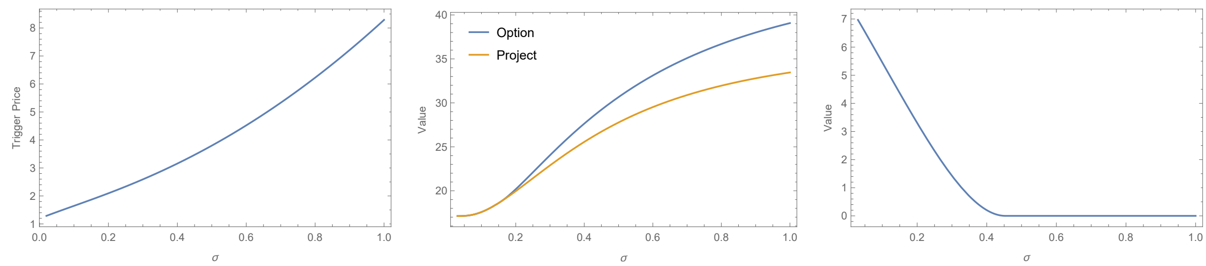
$$r = 0.04; \delta = 0.04; I = 10; R = 5.5; C_1 = 0.8; C_2 = 0.5;$$
$$\sigma = 0.2 \rightarrow P^* = 2.09443; \sigma = 0.4 \rightarrow P^* = 3.1559;$$
$$\sigma = 0.2 \rightarrow P_r^* = 2.4375; \sigma = 0.4 \rightarrow P_r^* = 23.9178$$
$$\sigma = 0.2 \rightarrow; P_c^* = 2.09443; \sigma = 0.4 \rightarrow P_c^* = 3.1559$$

Figure 4: Value of the options with different volatility values.

Changing the value of σ, in figure 4(a), we can see that it is positively correlated with the project and option value, as one case is more exposed to a higher profitability. This is easier to understand for the option, as we can choose not to invest on disadvantaged prices and therefore, the option is more valuable as we are exposed to higher profits. The projects value does not seem that obvious, as the higher the leverage the higher the gains and the losses, but since the negative part of the domain is limited by $P \geq 0$, the exposure to gains prevails. This previous condition, along with $F_i(P) \geq 0$ are the ones that grant more profitability for the option. The value of the project and the option become more and more steep as volatility increases. As for the P^* value increases to address for an higher exposure to possible down movements in P. Which on an NVP criteria seems a nonsense, because the project itself is more valuable, but we should only invest later. This is due to the fact that we are considering the optimum entry time.

The examples shown on figures 4(a) and 4(c) give two values for σ and show that the higher the dispersion on the values of P, the greater the project and option value can be. On the other hand, on Figure 4(b), one can see that testing with various values for the volatility on the Option to Renew yields different results, when accounting for the value of the project and its option, then in the Option to Invest. The higher the volatility the less the project of renewing seems attractive. The option also loses a lot of value but seems even more attractable when compared with the solo project of replacing, as the value of exercise goes to $P_r^* = 23.9178$. At a sum, there is a negative correlation. Again, in a less stable environment, the investor should be more tight and go passive, not exercising the option early in time, as the probability of changes in revenues are higher.

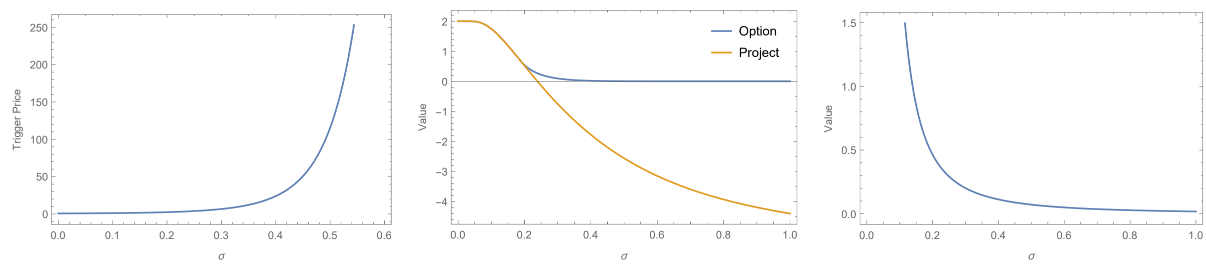
The behavior of the Option to Invest and Renew resembles the Option to Invest and comparing figure 7 to figure 5 one can arrive to that conclusion. The effect in the trigger prices is also positive, more volatility requires a higher price to enter, but the effect on the trigger price of the Option to Renew is the more significant, as with $\sigma = 0.4$ the trigger $P_r^* = 23.9178$ and $P_c^* \approx 3.16$.



(a) $P^*(\sigma)$ (b) $V_i(P0, \sigma)$; $F_i(P0, \sigma)$ (c) $F_i(0.6P^*(\sigma), \sigma) - V_i(0.6P^*(\sigma), \sigma)$

$r = 0.04$; $\delta = 0.04$; $C = 0.8$; $I = 10$; $P_0 = 1.88498$

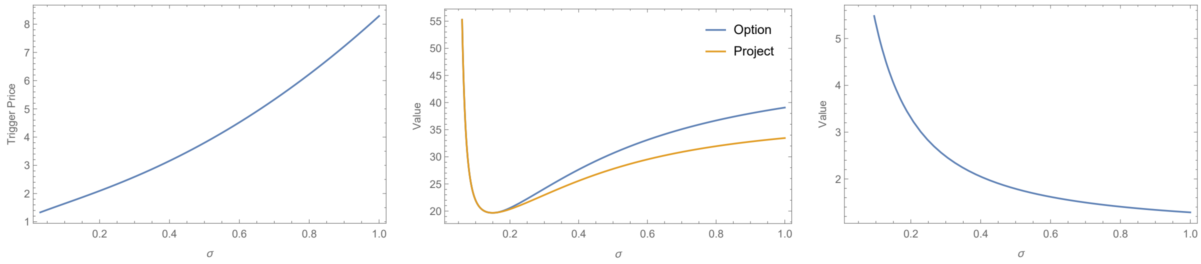
Figure 5: Effect of volatility on the Option to Invest - Trigger price, Option and project value and on the incremental option value.



(a) $P_r^*(\sigma)$ (b) $V_r(P_0, \sigma)$; $F_r(P_0, \sigma)$ (c) $F_r(0.6P_r^*(\sigma), \sigma) - V_r(0.6P_r^*(\sigma), \sigma)$

$r = 0.04$; $\delta = 0.04$; $R = 5.5$; $C_1 = 0.8$; $C_2 = 0.5$;$P_0 = 2.19375$.

Figure 6: Effect of volatility on the Option to Renew - Trigger price, Option and Project value and Incremental Option Value.



(a) $P_c^*(\sigma)$ (b) $V_c(P0,\sigma)$; $F_c(P0,\sigma)$ (c) $F_c(0.6P_c^*(\sigma),\sigma) - V_c(0.6P_c^*(\sigma),\sigma)$

$r = 0.04$; $\delta = 0.04$; $C_1 = 0.8$; $C_2 = 0.5$; $I = 10$; $R = 5.5$; $P_0 = 1.88498$.

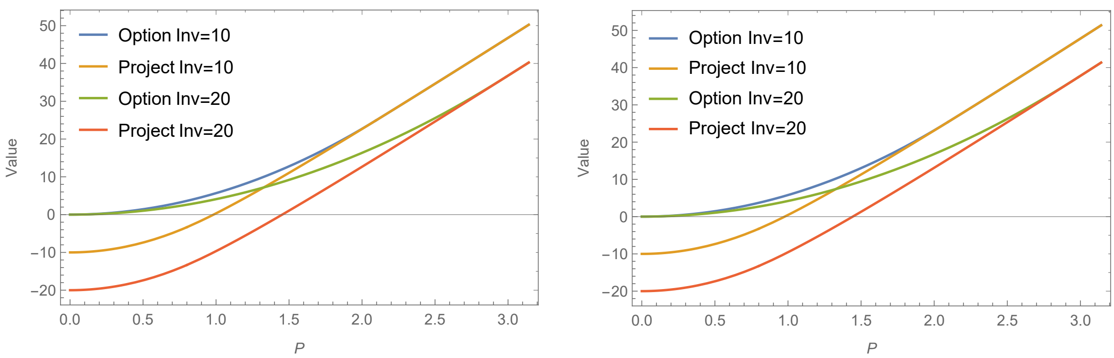
Figure 7: Effect of volatility on the Option to Invest and Renew - Trigger price, Option and Project Value and incremental option value.

Figure 5(a) expresses P^* in function of σ and establishes that positive correlation between both variables. Figure 5(b) confirms that the value of the project and option becomes more steep as volatility increases for volatility values higher than 0.2, although the difference between the project and option gets narrower as volatility increases, taking in account the increase on the trigger price. In other words, for a specified P value, as volatility increases the option turns more valuable than the project. Which in Figure 5(c) is shown not to be true for a fractions of the trigger price. The reason for that is that P^* increases very rapidly as volatility increases.

All plots listed on figure 6 show that volatility becomes more insignificant as it increases, either for its high exponential impact on the trigger price or for its null values of option value. In figure 4(b) it is visible the trigger price is higher as volatility increases. The next figure 6(a) illustrates that positive correlation between Pr^* and volatility. For values of $\sigma > 0.4$ the option would never be exercised, because the exposure to even higher uncertainty is meaningless as the value of the option is already 0, as seen in Figure 6(b), and because the trigger price would go up exponentially, as seen in 6(a). In this same figure, one can state how it negatively affect the value of the project and the option value matches and smooth pastes that loss, for a given value of P. At last, figure 5(c) shows how the value of the option changes along with the change of the trigger price, also proves that the incremental value of the option decreases as volatility increases. Which is contrary to the visible on the decreasing proximity in both lines of the previous graph, although justified by the high difference on the trigger price, as we are picturing the incremental value as a proportion of the trigger price. The Option to Renew, either simple or compounded, is by far the most sensible to a volatility changing environment, as the values of the projects turn almost insignificant and the trigger price increases rapidly, as seen on the scale used in figure 6(a).

The effect in the trigger price of the compounded option is positive, as described in figure 7(a), more volatility requires a higher price to enter. The effect on the trigger price of the Option to Renew is more significant, as with $\sigma = 0.4$ the trigger $P_r^* = 23.9178$ and $P_c^* \approx 3.16$, so we would invest only a bit later, but the renewal would be very improbable. As for the value of the project, the correlation is negative on lower volatility values and positive for $\sigma > 0.2$, at a given P value, as shown in 7(b). This represents the main difference between the behavior of volatility on the composed option and on the simple Option to Invest seen on figure 5(b). Volatility is negatively correlated with the value added by the option in comparison with the project, as seen in figure 7(c), conclusion applicable to every other option here studied.

4.2.2 *Investment - I*



(a) $V_i(P); F_i(P)$ (b) $V_c(P); F_c(P)$

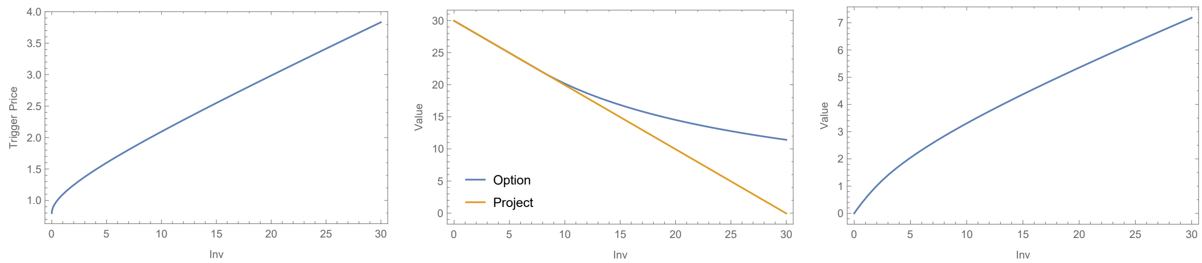
$r = 0.04; \delta = 0.04; \sigma = 0.2; C_1 = 0.8; C_2 = 0.5; R = 5.5; Pr = 2.4375;$

$I = 10 \rightarrow P^* = 2.09443; I = 20 \rightarrow P^* = 2.98564$

$I = 10 \rightarrow P_c^* = 2.09443; I = 20 \rightarrow P_c^* = 2.98564$

Figure 8: Value of the options with different investment values.

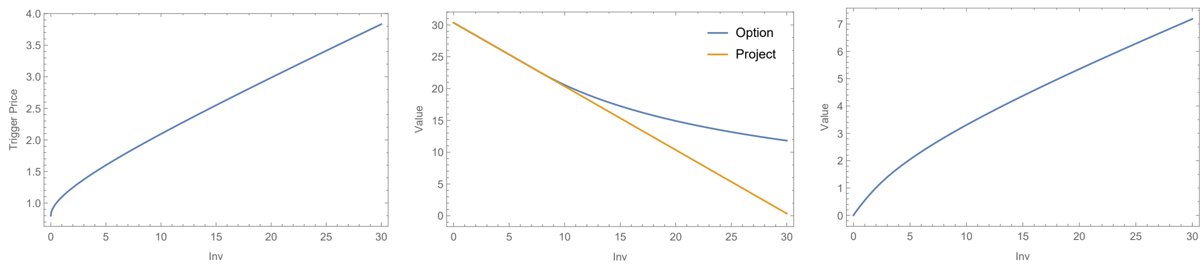
The higher the Investment, the lower the value of the project, although the much value comes with the option portfolio. Also the trigger price is higher. Choosing values as $I = 10$ and $I = 20$ yields us figure 8(a), all hypotheses are corroborated on the Option to Invest and Renew by plot on figure 8(b). These figures are very alike and show that the combined option's behavior is very similar to a simple Option to Invest, plus an increment of value that represents the value of the Option to Renew. The trigger price of the combined option seems to be achieved a little later, although the difference is not very significant as stated on the legend of figure 8. As the value of the investment increases, the project is less attractable, assuming all the market conditions stay the same. For a greater window of P's $[0, P^*]$, $V_i(P)$ is lower than $F_i(P)$, highly notable for lower P's. The value of the option decreases along with the value of the project, nevertheless it is even more attractable, as the gap between the option and the project is higher. Therefore, the difference on the trigger price, P^*, is highly noticeable too. As it is more costly to enter the market, the investor needs a more defensive margin, entering the market at higher P values, providing higher revenues to ensure that the operation is profitable.



(a) $P^*(I)$ (b) $V_i(P0, I)$; $F_i(P0, I)$ (c) $F_i(0.6P^*(I), I) - V_i(0.6P^*(I), I)$

$r = 0.04$; $\delta = 0.04$;$\sigma = 0.2$; $C = 0.8$; $P_0 = 1.88498$

Figure 9: Effect of the Investment on the Option to Invest - Trigger price, Option and Project value and incremental Option value.



(a) $P_c^*(I)$ (b) $V_c(P0, I)$; $F_c(P0, I)$ (c) $F_c(0.6P_c^*(I), I) - V_c(0.6P_c^*(I), I)$

$r = 0.04$; $\delta = 0.04$; $\sigma = 0.2$; $C_1 = 0.8$; $C_2 = 0.5$; $R = 5.5$; $P_0 = 1.88498$

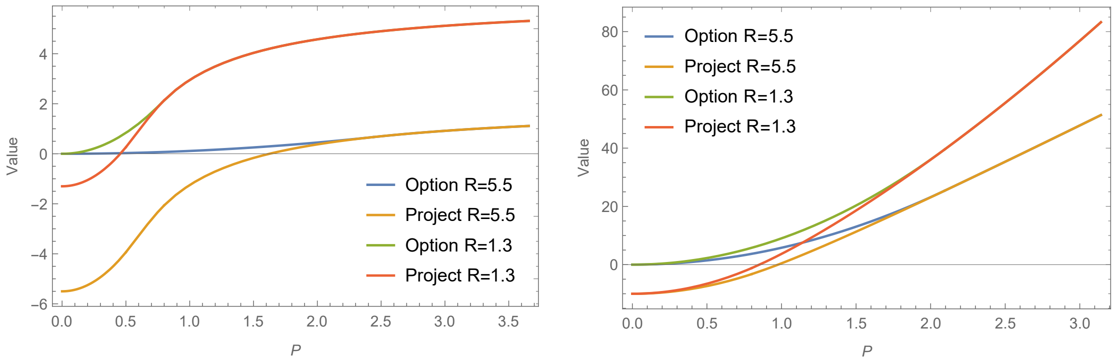
Figure 10: Effect of the investment on the Option to Invest and Renew - Trigger price, Option and Project Value and incremental option value.

The plot of P^* on the Investment, depicted in the previous figure 9(a), confirms that P^* and I are positively correlated. Its derivative is always positive, but also shows how this relation loses strength for higher I's, as the derivative goes to zero. What leads to a conclusion that for higher values of Investment, the time of entering the market is not that important. 9(b) shows how the value of the option and project split as the value of the investment increases, corroborating the hypotheses that the higher the Investment the more the option is valuable comparing to the project, as the intersection of the curves at P^* is done later. 9(c) represents the incremental value added by the option in comparison to the project, placing aside the change on the value of the trigger price and studies the value of the option and project at a constant fraction of P^*, whatever it might be. We can see that the higher the Investment, the more value we have added to the option and project. In this picture, we can see that the displacement between the value of the project and the option get higher with I, 9(c) represents the incremental value added by the option in comparison to the project at a given percentage of P^*.

As expected, the effect it has the on Option to Invest and Renew and on the trigger price is the same it had on the simple Option to Invest. P_c^* is positively correlated, so a higher investment price reflects as a higher P value needed to enter the market (figure 10(a)). And a higher investment cost means lower project value, and so are negatively correlated (figure 10(b)). For a fixed P value lower than P_c^*, the increment of value given by the option is positively correlated with the Investment, as expressed by figure 10(c)), as with an higher investment price the manager will chose not to enter the market.

As we are talking about an Option to Invest the behaviour is very likely the one expressed on a simple Option to Invest. As one can see all plots on figures 10 resemble plots on 9.

4.2.3 *Renewal cost - R*



(a) $V_r(P); F_r(P)$ (b) $V_c(P); F_c(P)$

$r = 0.04; \delta = 0.04; \sigma = 0.2; C_1 = 0.8; C_2 = 0.5; I = 10;$

$R = 5.5 \rightarrow P_r^* = 2.4375; R = 1.3 \rightarrow P_r^* \approx 0.79$

$R = 5.5 \rightarrow P_c^* = 2.09443; R = 1.3 \rightarrow P_c^* = 2.09443$

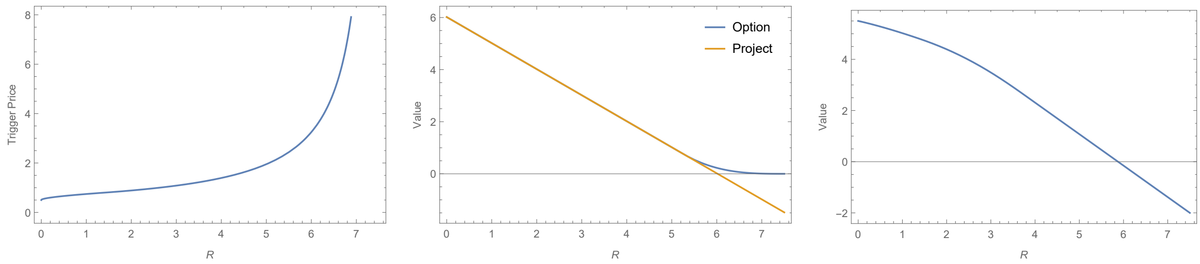
Figure 11: Value of the options with different renewal costs.

To counterpose to the $R = 5.5$, one chose a value of R below the boundary R^*, for example $R = 1.3$, and plotted both to see the difference in P_r^* and on the option value. The first one will have a trigger price which is a solution of $P_{r,3}^*$ and the second a solution of $P_{r,2}^*$.

Figure 11(a) shows that the higher the renewal cost, the lower the value of the project of replacing the aircraft and the lower the value of the option to later replace it. Again, as in the previous section 4.2.2, the higher the amount invested, the higher the trigger price, so with a higher renewal cost a higher value of P is chosen to exercise the replacement. The value of the project is also visibly punished by the increase on the price to replace and the options value loses power as the project is less valuable.

The higher the R the lower the value of the portfolio of the Option to Invest and Renew, as depicted in figure 11(b). As the exchange gets more expensive the replacement loses value and the option smooth pastes the project at lower values, losing value with it. This effect on the value is highly notable on higher P's, as both functions depart from the same origin as the condition on investing in the first place, remains unchanged. As the opportunity to renew gets more attractive the steepness of both functions increases.

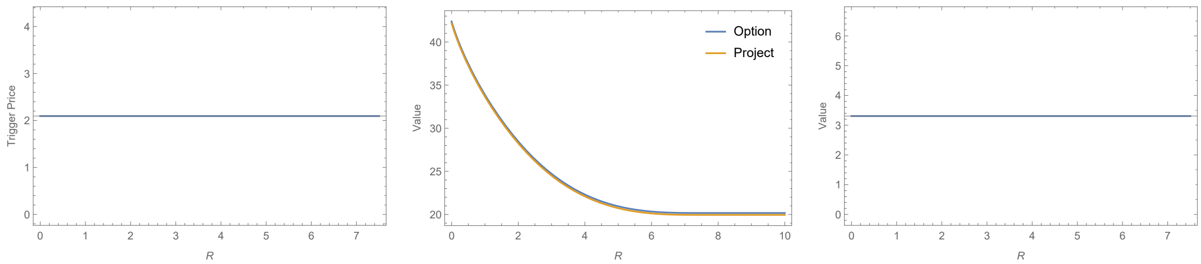
Either the simple and the compounded option are affected negatively by an increase on the renewal cost, although in different ways. In figure 11(a) the origin on the Y axis of project value is lower and in 11(b) the coefficient of growth is higher making the plot steeper.



(a) $P_r^*(R)$ (b) $F_r(P_0, R)$; $F_r(P_0, R)$ (c) $F_r(0.6P_r^*(R), R) - V_r(0.6P_r^*(R), R)$

$r = 0.04$; $\delta = 0.04$; $\sigma = 0.2$; $C_1 = 0.8$; $C_2 = 0.5$; $P_0 = 2.19375$.

Figure 12: Effect of the renewal cost on the Option to Renew - Trigger price, Option and Project value and Incremental Option Value.



(a) $P_c^*(R)$ (b) $F_c(P_0, R)$; $F_c(P_0, R)$ (c) $F_c(0.6P_c^*(R), R) - V_c(0.6P_c^*(R), R)$

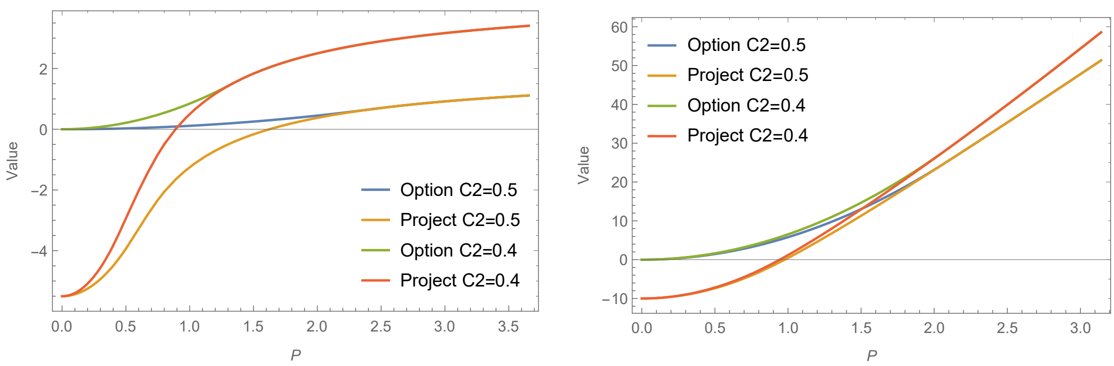
$r = 0.04$; $\delta = 0.04$; $\sigma = 0.2$; $C_1 = 0.8$; $C_2 = 0.5$; $I = 10$; $P_0 = 1.88498$.

Figure 13: Effect of the renewal cost on the Option to Invest and Renew - Trigger price, Option and Project Value and incremental option value.

In figure 12(a), a plot of the trigger price of the Option to Renew in function of the renewal that confirm its positive correlation and shows how it increases steeply on higher values of R. This solution P_r^* is a combination of $P_{r,2}^*$ for $R < R^*$ and $P_{r,3}^*$ for $R > R^*$, recall in this example $R^* = 1.4$. Plotting for value as in figure 12(b), we can see the negative correlation between the variable, R and the value of the project and option. Somewhere at an $R = 6$, the project turn unprofitable as the price to pay is too much for the efficiency gains in operating costs. At values lower than that, the option increases its value gradually. For values in between 5 and 6, although the renewal would still be profitable, the investor opts not to renew, as that residual gain in efficiency would not be enough to offset a possible down movement in price. Recall, that in this figure the smooth pasting condition is also visible. Figure 12(c) shows that the incremental value of the option is always negatively correlated with the renewal cost, for a certain proportion of P_r^*.

The trigger price, P_c^* is neutral to R, as seen in 13(a). Meaning that a change on the renewal cost of the opportunity to replace does not influence the time to enter the marker, which is counter intuitive, because one would expect to enter early if the possibility to renew would be more valuable. R is negatively correlated with the option and project to renew and invest, as seen in 13(b), which is also true for the simple option, as seen in the upper graph. A lower R increases the steepness of both functions exponentially. At values $R > 6$ the renewal has a constant residual value, as the replacement does not worth its cost. As the effect on P_c^* is null, figure 13(c) and figure 13(b) are true for any given P value and correctly represent each other, as the gap between the option and project is represented as 0 on the last plot. Meaning that the incremental value is constant independently of the renewal cost (figure 13(c)).

4.2.4 *Operating cost of the new aircraft - C_2*



(a) $V_r(P); F_r(P)$ (b) $V_c(P); F_c(P)$

$r = 0.04$; $\delta = 0.04$;$\sigma = 0.2$; $C_1 = 0.8$; $I = 10$; $R = 5.5$

$C_2 = 0.5 \rightarrow P_r^* = 2.4375; C_2 = 0.4 \rightarrow P_r^* = 1.33333$

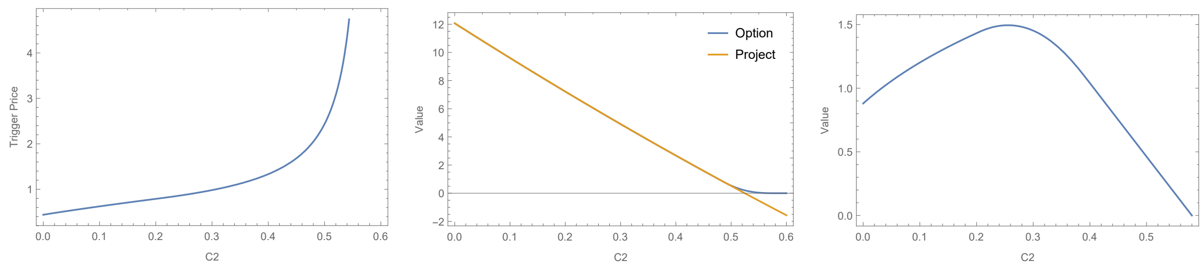
$C_2 = 0.5 \rightarrow P_c^* = 2.09443$; $C_2 = 0.4 \rightarrow P_c^* = 2.09443$

Figure 14: Value of the options with different operating costs for the new aircraft.

Using different values for C_2, yields us the following result presented on figure 14. As expected, lower operating costs add value to the renewal opportunity, as it converts in a higher efficiency gain, as $C_1 - C_2$ will be higher.

A higher C_2 turns the replacement less attractable, as so its option loses value with it, along with the compounded option. In figure 14(b), changing to a lower C_2 increases the steepness of both functions, as operations would become less expensive.

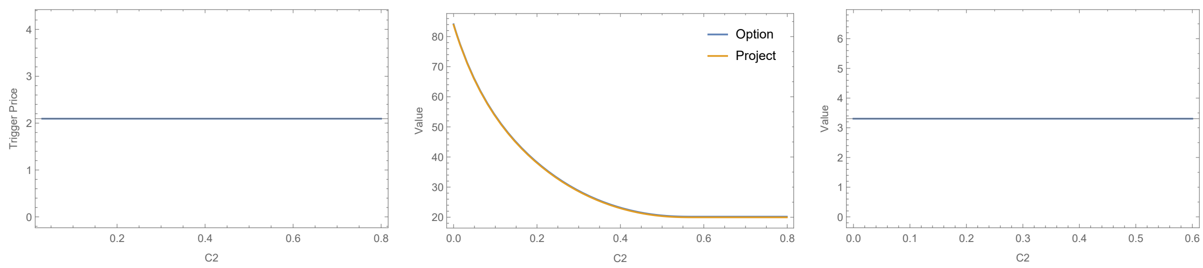
A higher C_2 pushes the value of the renewal to go down, as well as the value of the option. Take into account that a change in C_2 does not change the origin of the project and Option to Renew value, which maintains its value of 5, the value of the renewal cost. A lower value of C_2 get us closer to a lower volatility scenario, but the trigger price does not varies that much as in the volatility, conclusion taken by comparison of the previous 6(a) with the following figure 15(a). On the compounded option, although the functions behavior is different the effect of a lower C_2 is similar.



(a) $P_r^*(C_2)$ (b) $V_r(P_0, C_2)$; $F_r(P_0, C_2)$ (c) $F_r(0.6P_r^*(C_2), C_2) - V_r(0.6P_r^*(C_2), C_2)$

$r = 0.04$; $\delta = 0.04$; $\sigma = 0.2$; $R = 5.5$; $C_1 = 0.8$; $P_0 = 2.19375$.

Figure 15: Effect of the efficiency differential on the Option to Renew - Trigger price, Option and Project Value and incremental option value.



(a) $P_c^*(C_2)$ (b) $V_c(P_0, C_2)$; $F_c(P_0, C_2)$ (c) $F_c(0.6P_c^*(C_2), C_2) - V_c(0.6P_c^*(C_2), C_2)$

$r = 0.04$;$\delta = 0.04$;$\sigma = 0.2$; $C_1 = 0.8$; $I = 10$; $R = 5.5$; $P_0 = 1.88498$

Figure 16: Effect of the efficiency differential on the Option to Invest and Renew - Trigger price, Option and Project Value and incremental option value.

Figure 15(a) shows that C_2 is positively correlated with the exercise price of the Option to Renew. Figure 15(b) shows a negative correlation with the option and project to renew value. Its behaviour for a specific P value is very similar to the renewal cost, 12(b), there is a smooth pasting of both graphs and the investor should not exercise the option for $C_2 > 0.5$. Checking the value of the portfolio on a proportion of $P_r^*(C_2)$ in function of C_2, yields us interesting result, shown on figure 15(c), as an increase in low values of C_2 are positively correlated with the value created by the option, maybe that can be represented by the slow increase in the trigger price, but only an analytical analysis could conclude that.

The correlations between C_2 and the Option to Invest and Renew are the same as in the renewal cost, R, condition present on all plots on figures 16 and 13, as both these variables are from the embedded Option to Renew. On the combined option, the behaviour of these variables is very different then in the Option to Renew, conclusion taken from comparing figures 12 with 13 and 15 with 16.

Again, the change on the trigger price is null, as seen in 16(a), along with the increment of value to project at a given percentage of P_c^*, as seen in 16(c). As the trigger price remains almost unchanged, the option increment is correctly represented from figure 16(b), as both lines overlap each other.

4.3 SUMMARY

This section is a brief overview on how functions are influenced by changing variables. The reasons that set off these changes are always preset on the expression of the project value, as it is the project value that conducts the option value. At the boundary condition, $P = P^*$, an analytical analysis can get us to the same results as the shown bellow, although as variables depend on each other, it turns out very difficult to check the partial derivatives signal, being unable to establish a positive or negative trend.

With different values of investment (I), the price of starting operations is lower, therefore the Y interception is lower, each graph literally moves down, and the project loses the same value for any P.

With different values for the renewal cost (R), the project value gets less steep and therefore, the higher the value of P, the more value is lost on the replacement of the aircraft, as expected. This is valid for the Option to Invest and Renew and in the

Option to Renew, as the renewal cost acts as the investment paid up front, moving the graph down, just as the investment on investing options.

The volatility affects the slope of the lines, it has more effect for lower P values, as the graphs tend to a straight line fairly quickly.

As for the efficiency gained by the renewal, represented by C_2, it also affects the steepness of the functions for each of the options with renewal.

These effects are visible and explained on the graphs of the value of the project and option in function of these variables. In table 2, it is shown the sum of all the correlations between each parameter, in each option, and the:

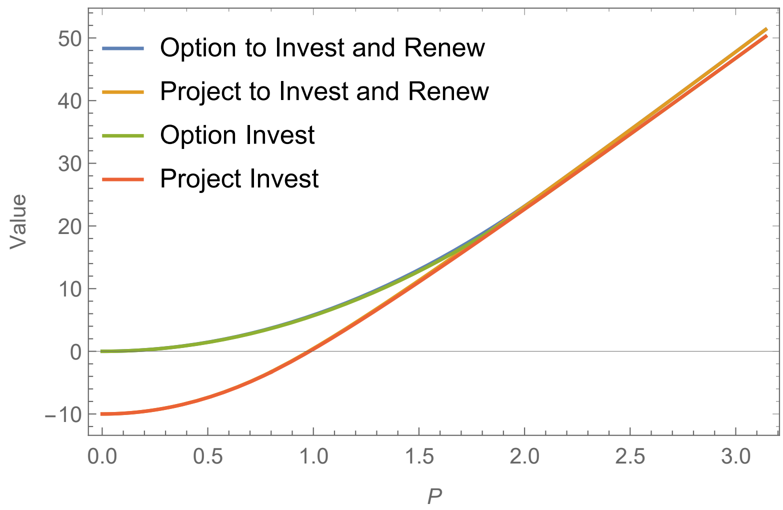
- P^* - Trigger price of each option.
- $F(P_0)$ - Value of the project and option for a given P value.
- $F(0.6P^*) - V(0.6P^*)$ - Value added by the option compared to the project at a given fraction of the trigger price. [1]

Parameter	Description	P^*	$F(P_0)$	$F(0.6P^*) - V(0.6P^*)$
Option to Invest				
I	Investment	+	-	+
σ	Volatility	+	+	-
Option to Renew				
R	Renewal cost	+	-	-
σ	Volatility	+	-	-
C_2	Operational cost	+	-	$\pm$
Option to Invest and Renew				
I	Investment	+	-	+
R	Renewal cost	0	-	0
σ	Volatility	+	$\pm$	-
C_2	Operational cost	0	-	0

Table 2: Summary of Sensibility Analysis.

It is also interesting to compare the value of the simple Option to Invest, against the value of the combined option, shown on the next figure 17.

1 Bear in mind that the trigger price itself depends on each one of these parameters.



$r = 0.04$; $\delta = 0.04$; $\sigma = 0.2$; $C_1 = 0.8$; $C_2 = 0.5$ $I = 10$; $R = 5.5$; $P_c^* = 2.09443$;$P^* = 2.09443$

Figure 17: Value added by the embedded Option to Renew.

As expected, the value of the project with both options included is higher than the simple project to invest. As the renewal project gets more attractable, with lower renewal costs and operational costs, the slope of the combined option increases, meaning the value added by the options, represented by the gap between both projects lines, gets greater and greater. The trigger price for the investment is unaffected, which is non intuitive as said before. On the other hand, the trigger price for the renewal is lower.

5

APPROACH TO REALITY AND MOTIVATION FOR FUTURE WORK

In this work, each project is seen as a single or combined American Option, on infinite lived assets. In reality, each option can be exercised anytime during the lifetime of the aircraft, which is correctly represented in the model. The options are combined so that only meaningful options overlay, like we can only replace the aircraft after investing in the first place. Although, obviously, aircraft only last some 30 years, instead of those infinite series created. On the other hand, the difference is not that significant since, after n=30, the value of the option is already totally represented. Also, this difference can be seen as a salvaging value offsetting the premium that results on the infinite lived model. Taxes and depreciations are disregarded.

Each options takes effect immediately as exercised, it is assumed no time to put in practice a decision taken by managers. A datum can be placed to make the model more representative of reality, assuming we know the time it takes to put into place the strategy to take advantage to of those opportunities.

The parameters C, C_1, C_2, r, δ, σ, I, R are assumed to be constant, when in fact they are not. Obviously, all these values change on real time, but adding more stochastic variables would turn the model very complex to model. Therefore, P, is the only stochastic variable, which represents the revenues of operating the aircraft. Transporting it into a passenger operation, P would be the total income from tickets, or a ticket price times quantity, Q, which is not present on hereby models. P is the driving variable for revenues and C is the fixed cost of operating the aircraft. For them to be even and comparable, I would use Revenue per Available Seat Mile (RASM) and Cost per Available Seat Mile (CASM) and introduce a quantity, Q, for application into different aircraft and on different occupation rates.

As we are talking about a cyclical industry, the possibility to suspend and reactivate operations would seem to be very attractive to address fluctuations in prices. In fact, the option to suspend a project does not seem to have value in real life when looking to at the airline as a hole, as it is very rare for an airline to suspend operations and then reactivate again, as if nothing has happened. We have sorts of cases at which airlines suspended operations and ended up reporting bankruptcy a few weeks later. Though, if we see each aircraft like a project, it is more frequent to see a suspend of operation, because sometimes airlines prefer to pay the airport the parking of an aircraft, instead of operating it. Which happened recently to an A330 from Azores Airlines that ended up parked in Oporto and later in France. Unsuccessfully, its reactivation was never made possible and HiFly will, in due date, start to operate it.

A more general and actual example, is the epidemic COVID-19 that restricted the flow of passengers, either by travel restrictions, countries locked down or just goodwill or fear from the passengers. That caused a dramatic decrease of 80% around China and 4.4% on a global stand point on February 2020. In March, the pandemic's imposed stop was felt globally and commercial traffic was down 27.7% below 2019 levels (Petchenik (2020)). Various airlines have cut capacity, reduced the frequency of certain flights, suspended routes, parked the aircraft and granted their employees unpaid leaves or layoffs. Some airlines even ceased its existence, since the lessors came up after the aircraft as the operator was incapable of keeping up with the pay. To prevent unprofitable operations, airlines can stop operations or at least some of them, as some restrictions that oblige carriers to fly are also being cancelled, by authorities such as the FAA and EASA. These airlines exercised their options to contract and suspend and so, the value salvaged on operating costs should also be accounted during the investment evaluation process.

In this investments opportunity, the cost of operating the aircraft encloses a combination of factors, such as: fuel; human resources; leasing costs; airport fees; maintenance and many others. The difficulty of valuating all these parameters under uncertainty is very high, so the degree of accuracy is sacrificed. A parked aircraft also has its costs, as the owner will have to pay for the parking lot, the maintenance roster, maintenance flights and personal, so the suspension comes with costs too. These costs depend on the type of parking practiced by the airlines, depending on the time that the airline pretends to store the aircraft. An active storage is supposed to be a short time park, as the airline expects to need the aircraft in a nearby time frame. This practice is costly to maintain the aircraft, although the aircraft is almost ready to start operating. A deep storage suggests a long time frame, around 6 months to 2 year, possibly the aircraft is taken to a boneyard, with low maintenance costs but it can still be put into operations, although its recuperation is more costly and time consuming, as some parts may have been removed in order to avoid deterioration.

Hence, in this work the suspension was modeled but not presented. It would be interesting to further study the option to suspend, along with the options studied here. The option would encompass a possibility to invest with two options embedded, the option to suspend and the Option to Renew. This requires more project valuations, as we would be creating various project: waiting to invest and invested; old aircraft and new aircraft; and active and suspended projects. So, going from condition to condition and assuming only a 3 step process can produce a tree of branches that make sensible decisions in a real-life project where:

- The investor starts with no aircraft, only with the investment opportunity;
- After investing can suspend operations, keep operating the old aircraft or replace it for a more efficient one;
- After having the new aircraft can suspend operations or keep operating.

After creating the project values of each, the combination of passing through states would produce various boundary conditions, which would lead to a system of 4 equations, where various coefficients would have to be discovered together for us to have accurate trigger prices. This possibility would better describe a real life branch of options, but demands an exhaustive workload. Further researches may specifically focus on this matter and produce an interesting piece of study.

6

CONCLUSION

This dissertation shows the value added by options, how they influence project management and why they should be accounted during evaluation. The purpose of creating a model with two options combined, evaluated simultaneously, applied to the aviation industry was accomplished and helps clarify the decision making.

The trigger price for the start of operations was found in function of the stochastic variable and varies according to the other variables. The higher the investment and the volatility, the higher the trigger price, in order to address for future possible down movements in price. On the other hand, the renewal cost and the operational cost of the new aircraft do not change the optimum time to invest. The value of the project and the option are negatively correlated with the investment, renewal cost and operational cost. Correlation with volatility is non specified, as it is negatively correlated for lower values of volatility and positively for higher values. These conclusions are applicable for the Option to Invest and Renew, whose correlations are not always true for the simple options.

Both options combined are relevant to such industry, as timing the market and evolving with it are possibly the main causes for success. Extensions to this model are perfectly possible and would probably produce very interesting conclusions. Although the introduction of other options, as the option to suspend and activate with costs, would more precisely modulate the reality of the market, they would come at a high cost. Moreover, the options explored in this dissertation are of great value for the air transport industry and again prove, that a standard approach neglects the value of any future opportunities.

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